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# Comments on Limits of Space-Times[1]

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A coordinate-dependent investigation of limits of space-times is given. Different kinds of limits are found and analyzed. A criterion selecting limits which appear to be nothing but "technical tricks" is proposed. Finally, a method for generating new limits from a given family of metrics is examined.

## 1. INTRODUCTION

In the study of general relativity, it frequently occurs that solutions of Einstein's field equations may be collected into families of space-times depending on some free parameters. This enables one to consider limits of space-times when a given parameter $\lambda$ approaches a certain fixed value $\lambda_0$ (e.g., $\lambda_0 = 0$). In this context, one usually understands limits in a strictly coordinate-dependent manner. Owing to this fact, one of the most surprising results is, perhaps, the possibility of finding different space-times as limit.

A precise formulation of this problem was given by Geroch (Geroch, 1969). There, the possibility of finding new solutions of Einstein's equations as limit of known solutions was pointed out. This is in close connection with the formidable problem of determining *all* the limits of an assigned metric or, otherwise, of deciding whether a given class of solutions is closed or not.[2] Insights may be obtained by understanding how the limit operation works.

[1]This work was carried out under the auspices of the National Group for Mathematical Physics of C.N.R.

[2]A family of solutions of Einstein's equations is called a *closed class* if it contains all its limits (Geroch, 1969).



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The main purpose of this note is to describe, by means of a critical analysis of the actual algorithm, how different space-times may be obtained as limits of a given family of metrics as $\lambda \to 0$. In this context, we enunciate a property in order to characterize limits which are nothing but "technical tricks"; as a further result, the present investigation provides a method for generating limits from a given family of metrics.

## 2. ANALYSIS OF THE LIMIT PROCEDURE

In the following it is useful to adopt a coordinate-dependent approach as this lies special stress on the possibility of finding different space-times as $\lambda \to 0$. Although this approach involves only local properties, no generality is lost in view of the theorem that "the global (maximal) limit of a family of space-times is uniquely determined by the local knowledge of the limit" (Geroch, 1969).

Accordingly, consider the family of vacuum metrics

$$ds^2 = -\left(\lambda^2 - \frac{\beta^2}{r}\right)dt^2 + \left(\lambda^2 - \frac{\beta^2}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \lambda^{-2}\sin^2\lambda\theta\, d\phi^2) \tag{2.1}$$

In particular, putting $\lambda = 1$, $\beta^2 = 2m$, we obtain the Schwarzschild vacuum metric. On the other hand, the choice $\lambda = 0$, $\beta^2 = 2$ yields the Kasner metric (Kasner, 1921). Now, assuming $\lambda \neq 0$, the following coordinate transformation,

$$r = \lambda\rho, \qquad t = \tau/\lambda, \qquad \theta = \xi/\lambda \tag{2.2}$$

makes the metric (2.1) into the form

$$ds^2 = -\left(1 - \frac{\beta^2}{\lambda^3\rho}\right)d\tau^2 + \left(1 - \frac{\beta^2}{\lambda^3\rho}\right)^{-1} d\rho^2 + \rho^2(d\xi^2 + \sin^2\xi\, d\phi^2) \tag{2.3}$$

Once again, we obtain the usual form of the Schwarzschild metric by putting $\beta^2/\lambda^3 = 2m$.

The metric (2.3) has no limit as $\lambda \to 0$. Nonetheless, it is possible to restore the parameter $\lambda$ into its original position by means of the inverse of the transformation (2.2) thus obtaining the Kasner metric as $\lambda \to 0$. We remark this is exactly Geroch's example. Therefore, in order to obtain the required interpretation of the limit process, it is necessary to analyze the role of the preliminary coordinate transformation.

To this end, we notice that, in the previous example, the Schwarzschild metric is the most general line element which can be obtained from (2.1) by choosing $\lambda \neq 0$ and $\beta \neq 0$. Under this assumption, it turns out that the coordinate transformation (2.2) has essentially packaged the two parameters $\lambda$ and $\beta$ in the only one $\beta^2/\lambda^3$ which assumes the value $2m$ in the Schwarzschild metric. According to this view, it is evident that the inverse of (2.2) transfers the parameter $2m$ to the position of $\lambda$ in (2.1).

In general, consider a given line element as member of a suitable family of metrics, the members of which depend on the choice of some free parameters (i.e., integration constants): of course some parameters are *already* fixed in the original metric. Then, the coordinate transformation has essentially the role of restoring a given parameter $\lambda$ in the position of one of the fixed parameters. So, in general, the subsequent limit $\lambda \to 0$ yields a different member of the family. This enables us to conjecture that the metric obtained as limit of a family $\{g(\lambda)\}$ as $\lambda \to 0$ is "less general" than any member $g(\lambda)$. This argument is also supported by the existence of hereditary properties.[3] In fact, consider a scalar quantity constructed only from the metric tensor and its derivatives [e.g., the 14 second-order Petrov's scalars (Petrov, 1969)] and suppose that it vanishes for every member $g(\lambda)$ of the family. In view of the smooth behavior of $g(\lambda)$ as $\lambda \to 0$, it is an easy matter to show that this scalar vanishes also for the limit metric. In general, however, a scalar quantity of this kind can, at the most, decrease to zero as $\lambda \to 0$. In particular, this implies that the limit metric is at least as algebraically specialized as any member of the family.

As a matter of fact, the previous discussion does not exhaust the problem. To this end, consider the Kerr metric in the usual Boyer and Lindquist coordinates (Boyer and Lindquist, 1967)

$$ds^2 = \rho^2(dr^2/\Delta + d\theta^2) + (r^2 + a^2)\sin^2\theta\, d\phi^2 - dt^2 + (2mr/\rho^2)(a\sin^2\theta\, d\phi - dt)^2 \tag{2.4}$$

where $\rho^2 = r^2 + a^2\cos^2\theta$ and $\Delta = r^2 - 2mr + a^2$. It is well known that, in these coordinates, the limit $a \to 0$ yields the Schwarzschild metric. However, in terms of the coordinates

$$x = r - a^{-1}, \qquad \xi = \theta/a \tag{2.5}$$

[3]We recall that "a property of space-time is called *hereditary* if, whenever a family $\{g(\lambda)\}$ of space-times have that property, all the limits of this family also have the property" (Geroch, 1969).

the limit $a \to 0$ now gives the Minkowski space-time. We notice that the transformation (2.5) does not fall within the previous interpretation: in fact, choosing

$$x = r - \alpha^{-1}, \qquad \xi = \theta/\alpha \tag{2.6}$$

$\alpha$ being an unspecified parameter, the subsequent limit $\alpha \to 0$ yields nevertheless the flat space-time. In the present case, the transformation (2.5) looks like a pure technical trick: the misleading choice $\alpha = a$ has essentially the role of avoiding the introduction of an unphysical parameter.

In this connection, we give a property which enables us to single out limits of this kind. To this end, we schematize the limit procedure as follows. Consider a family of metric $\{g(\lambda)\}$ depending on the parameter $\lambda$; suppose that $\{g(\lambda)\}$ has no limit as $\lambda \to 0$ but there exists a class of diffeomorphisms $\{\psi_\alpha: V_4 \to V_4\}$, singular in $\alpha = 0$, such that the family $\{\psi_{\lambda *} g(\lambda)\}$ admits limit as $\lambda \to 0$.[4]

We say that a limit is not a pure technical trick if it satisfies the following:

*Property 2.1.* Let $\{g(\lambda, \beta)\}$ be the two-parameter family of metrics whose elements are defined as $g(\lambda, \beta) = \psi_{\beta *} g(\lambda)$. Then (i) $\{g(\lambda, \beta)\}$ does not admit limit as $\beta \to 0$, and (ii) there exists a relation $\beta = \beta(\lambda)$ such that $\{g(\lambda, \beta(\lambda))\}$ has limit as $\lambda \to 0$.

It is a straightforward matter to check that our first example satisfies Property 2.1, while the second one does not.

As a final remark, we analyze the particular aspect concerning the signature of the metric. In this connection, let us consider the Kerr metric (2.4). By means of the transformation

$$\cos\theta = \mu/a, \qquad \phi = a\psi \tag{2.7}$$

after some algebraic manipulations, the limit $a \to 0$ now gives

$$ds^2 = (r^2 + \mu^2)(dr^2/\Delta - d\mu^2/\mu^2) - (r^2 + \mu^2)^{-1} \times \left[ \mu^2 (dt - r^2 d\psi)^2 + \Delta (dt + \mu^2 d\psi)^2 \right] \tag{2.8}$$

where $\Delta = r^2 - 2Mr$.[5] The metric (2.8) has signature $-2$ unlike the Kerr metric (2.4). Nonetheless (2.8) is a solution of Einstein's vacuum equations

[4]Here, $V_4$ is the space-time manifold and $\phi_{\alpha *}$ is the map acting on tensor fields induced by the diffeomorphism $\phi_\alpha$ (Sternberg, 1964).

[5]This example may be deduced by Carter's family of line elements (Carter, 1973) using the technique just described.

with Lorentz signature: hence it turns out to be again a "physically" acceptable metric.

More pathological situations deserve further consideration. For example, acting with the transformation

$$\mu = (\lambda - 1)\sin\xi, \qquad \psi = \phi/(\lambda - 1) \tag{2.9}$$

on (2.3), we get as limit $\lambda \to 1$ a metric with signature 0. Moreover, the further transformation

$$\mu = \nu e^{(\beta-1)y}/(\beta - 1), \qquad \psi = -\left[(\beta-1)^2\nu^{-2} - 1\right]^{1/2} e^{-(\beta-1)y} \tag{2.10}$$

and the subsequent limit $\beta \to 1$ restore the Lorentz signature $-2$. Therefore, this shows that it could be convenient to consider on the same footing all the metrics regardless their signature, in connection with the problem of finding new solutions as limits of known solutions.

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